TWO PROBLEMS RELATED TO PRESCRIBED CURVATURE MEASURES

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ABSTRACT. Existence of convex body with prescribed generalized curvature measures is discussed, this result is obtained by making use of Guan-Li-Li's innovative techniques. In surprise, that methods has also brought us to promote Ivochkina's C^2 estimates for prescribed curvature equation in [12, 13].

1. INTRODUCTION AND MAIN RESULTS

Curvature measure plays fundamental role in the theory of convex bodies, which is closely related to the differential geometry of convex hypersurfaces and integral geometry. It has been extensively studied, see Schneider's book [18]. As Guan, Li and Ma etc. [8, 9] and their references. Here we give the interpretation of the problem from point of partial differential equation for example see [9]. We view $\mathbb{M}$ as a graph over $\mathbb{S}^n$, and write $X(x) = \rho(x)x$, $x \in \mathbb{S}^n$, $\forall X \in \mathbb{M}$. Therefore the problem of prescribed (n-k)th curvature measure can be reduced to the following curvature equation:

$$\sigma_k(A) = \frac{f\rho^{1-n}}{\sqrt{\rho^2 + |\nabla\rho|^2}}, \tag{1.1}$$

where $f > 0$ is the given function on $\mathbb{S}^n$. Moreover, equation (1.1) can be expressed as differential equations on radial function ρ and position vector X,

$$\sigma_k(A) = |X|^{-(n+1)} f\left(\frac{X}{|X|}\right)\langle X,\ \nu\rangle \triangleq \phi(X)\langle X,\ \nu\rangle, \tag{1.2}$$

where ν is the unit outer normal of $\mathbb{M}$, and $\lambda = (\lambda_1, \cdots, \lambda_n)$ is the principal curvature of M at point X,

$$\sigma_k(A) = \sigma_k(\lambda) = \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \lambda_{i_1}\cdots\lambda_{i_k}.$$

Alexandrov problem is the zero order curvature measure, which can also be considered as a counterpart to Minkowski problem. Its regularity in elliptic case was proved by Pogorelov [17] for $n = 2$ and by Oliker [16] for higher dimension case. The degenerate case was obtained by Guan-Li [7].

Following ideas from [1, 13, 21] etc., let us define the $k-$admissible hypersurfaces:

Definition 1.1. For $1 \leqslant k \leqslant n$, let Γ_k be a cone in $\mathbb{R}^n$ determined by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_l(\lambda) > 0,\ l = 1, 2, \cdots, k\}.$$

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A smooth hypersurface $\mathbb{M}$ is called $k-$admissible if $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Gamma_k$ at every point $X \in \mathbb{M}$.

There is a difficulty issue around equation (1.2): the lack of some appropriate a priori estimates for admissible solutions due to the appearance of gradient term at right side. That problem has been open for many years [4]. More recently, Guan-Li-Li[10] have obtained the C^2 a priori estimates for the admissible k-convex starshaped solutions to prescribing (n-k)th curvature measures for $1 \leqslant k < n$.

In this paper, we are interested in to consider the following problems

$$\sigma_k(A) = \langle X,\ \nu\rangle^p \phi(X), \tag{1.3}$$

where $2 \leqq k \leqq n$.

Our first motivation is from the existence of convex body with prescribed curvature measures. Equation (1.3) is the equation of prescribing $(n-k)$th curvature measure for $p = 1$. In particular, Guan-Li-Li[10] has given a open problem for most general problem in remark 3.5. our result may implies that their conjecture is correct. Caffarelli-Nirenberg-Spruck[1] had been considered some kind of curvature equation including (1.3) for $p = 0$, their C^1 estimates depends on barrier conditions. However, that is not case for our problem. Thus, we consider (1.3) for $p \neq 0$. Moreover, we can obtain C^1 estimates for a class of curvature equations including (1.3) and quotient curvature equations, its idea is from [9]. Now, we can state the main theorem.

Theorem 1.2. *Suppose $\phi(X)$ is a smooth positive function in $\mathbb{M}$, $2 \leqslant k \leqslant n$, $0 \neq p \leqslant 1$. Then there is a unique smooth admissible hypersurface $\mathbb{M}$ satisfying (1.3).*

Our second motivation is to generalize Ivochkina's C^2 estimates for prescribed curvature equation in [12, 13] by making use of those methods. Ivochkina[12, 13] had considered the generalized type of curvature equation, see also [2, 3, 4, 19, 21, 22] and their references,

$$\sigma_k(A) = \sigma_k(\lambda) = \phi(x, g, Dg), \tag{1.4}$$

where A and $\lambda = (\lambda_1, \cdots, \lambda_n)$denote respectively the second fundamental form and the principle curvatures of the graph

$$\mathbb{M} = \{(x, g(x))|x \in \Omega\}.$$

For doing C^2 estimates of (1.4), She needed her condition (1.5) in [12](see also (8.28) in [13]),which is

$$k\frac{\partial^2 \chi^{1/k}}{\partial p^i \partial p^j}\xi_i\xi_j \geqslant -\frac{\chi^{1/k}}{2\sqrt{n}(1 + (\max|p|)^2)}\xi^2, \quad \xi \in R^n, \tag{1.5}$$

where $\chi(x, g, p) = \phi(x, g, p)(1 + |p|^2)^{\frac{k}{2}}$. We consider mainly a kind of model from Takimoto[20], which is also seen as translating solution of curvature flow.

$$\sigma_k(\lambda) = \frac{H(x, g)}{(1 + |Dg|^2)^{\frac{q}{2}}}, \tag{1.6}$$

Ivochkina's conditions (1.5) in [12] needs $q \leqslant 0$. However, we can generalize Ivochkina's C^2 estimate to $q \leqslant 1$.

Theorem 1.3. *Suppose $g \in C^4(\Omega)\bigcap C^2(\overline{\Omega})$ is an admissible solution of* (5.2) *for $q \leqslant 1$. Then the second fundamental form A of graph u satisfies*

$$\sup_{\Omega}|A| \leqslant C\left(1+\sup_{\partial\Omega}|A|\right), \tag{1.7}$$

where C depends only on n, $\|g\|_{C^1(\Omega)}$, and $\overline{\Omega}\times[\inf_{\partial\Omega} g,\ \sup_{\partial\Omega} g]$.

This paper is organized as follows: The C_0 and C_1 bounds and some elementary formulas were listed in section 2, the important C^2-estimates are derived in section 3, which is by using Guan-Li-Li's innovative methods. In the last section, we can generalize Ivochkina's C^2 estimates for prescribed curvature equation in [12, 13].

2. Some elementary formals and C^0-C^1 boundness

The standard basis of $\mathbb{R}^{n+1}$ will be denoted by $\mathbb{E}_1, \mathbb{E}_2, \cdots \mathbb{E}_{n+1}$, and the components of the position vector X in this basis will be denoted by $X_1, X_2, \cdots, X_{n+1}$. We choose an orthonormal frame such that $e_1, e_2, \cdots e_n$ are tangent to $\mathbb{M}$ and ν is normal.

The second fundamental form of $\mathbb{M}$ is given by

$$h_{ij} = \langle D_{e_i}\nu, e_j\rangle, \tag{2.1}$$

and some fundamental formulas are well known for hypersurfaces $\mathbb{M} \in \mathbb{R}^{n+1}$ as [9].

Lemma 2.1. *For any $i, j, l, m = 1, \cdots, n$,*

$$\nabla_i\nabla_j X = -h_{ij}\nu, \tag{2.2}$$

$$\nabla_l\nabla_l h_{ij} = \nabla_i\nabla_j h_{ll} + h_{lm}h_{ij}h_{ml} - h_{ll}h_{im}h_{mj}, \tag{2.3}$$

$$\nabla_i\langle X,\ \nu\rangle = h_{il}\langle\nabla_l X,\ X\rangle, \tag{2.4}$$

$$\nabla_j\nabla_i\langle X,\ \nu\rangle = \nabla_l h_{ij}\langle\nabla_l X,\ X\rangle + h_{ij} - uh_{im}h_{jm}. \tag{2.5}$$

Owing to $\mathbb{M}$ be compact, the C^0 estimates is obvious as Lemma 2.2 in [9].

Lemma 2.2. *For any compact hypersurface $\mathbb{M}$ satisfying* (1.3)*, there are two positive constant C_1, C_2 such that*

$$C_1(n, k, \min_{\mathbb{S}^n} f) \leqslant \min_{\mathbb{S}^n}|X| \leqslant \max_{\mathbb{S}^n}|X| \leqslant C_2(n, k, \max_{\mathbb{S}^n} f).$$

We have the following gradient estimate for general curvature measure equation which included (1.3) and curvature quotient equations. As in Guan-Lin-Ma[9], the result will be obtained without any barrier condition that was imposed in [1]. Moreover, our result holds for any non zero p, i.e. $0 \neq p \in (-\infty, +\infty)$.

$$F(A) = f(\lambda) = \langle X,\ \nu\rangle^p\phi(X) \tag{2.6}$$

Lemma 2.3. *If $\mathbb{M}$ satisfies* (2.6) *for F with homogeneous of degree $t > 0$, and for $0 \neq p \in (-\infty, +\infty)$, then there exist a constant C depending only on n, t, p, $\min_{\mathbb{S}^n}\phi$, $|\phi|_{C^1}$ such that*

$$\max_{\mathbb{S}^n}|\nabla\rho| \leqslant C \tag{2.7}$$

Proof. We use the method of Guan-Lin-Ma[9]. The gradient bound is equivalent to $u = \langle X,\ \nu\rangle \geqslant C > 0$ if the lower and upper bound of the solution holds.

Setting

$$P(X) = \gamma(\frac{|X|^2}{2}) - \log\langle X,\ \nu\rangle,$$

where the function $\gamma(s)$ is to be determined.

Assume $P(X)$ attains its maximum at point $X_0 \in \mathbb{M}$. We choose the smooth local orthonormal frame $e_1, \cdots, e_n \in T_{X_0}\mathbb{M}$ such that

$$\langle X,\ e_i\rangle = 0,\ i \geqslant 2$$

Thus $|X|^2 = \langle X,\ e_1\rangle^2 + \langle X,\ \nu\rangle^2$. If $\langle X,\ e_1\rangle^2$ is also zero, then $|X|^2 = \langle X,\ \nu\rangle^2$, then the bounded from below of $\langle X,\ \nu\rangle$ is from the bound of $|X|$. We now consider $\langle X,\ e_1\rangle^2 > 0$, one has

$$0 = \nabla_i P(X) = \gamma'\langle X,\ e_i\rangle - \frac{h_{im}\langle X,\ e_m\rangle}{\langle X,\ \nu\rangle},$$

which implies

$$h_{11} = \gamma'\langle X,\ \nu\rangle, \qquad h_{1i} = 0, \quad i \geqslant 2. \tag{2.8}$$

It is easy to know that we only fix e_1 in above process of choosing local orthonormal frame field $e_1, \cdots, e_n$, here we adjust $e_2, \cdots, e_n$, such that $A = [h_{ij}]$ is diagonal at X_0, and

$$\begin{aligned}
0 \geqslant\ & F^{ii}\nabla_i\nabla_i P = \gamma'' F^{11}\langle X,\ e_1\rangle^2 + \gamma' F^{ii}[\delta_{ii} - h_{ii}\langle X,\ \nu\rangle] \\
& - \frac{\langle X,\ e_1\rangle}{\langle X,\ \nu\rangle} F^{ii}\nabla_1 h_{ii} - \frac{F^{ii}h_{ii}}{\langle X,\ \nu\rangle} + F^{ii}h_{ii}^2 + F^{11}\frac{h_{11}^2\langle X,\ e_1\rangle^2}{\langle X,\ \nu\rangle^2} \\
& = [\gamma'' + (\gamma')^2]F^{11}\langle X,\ e_1\rangle^2 + \gamma' F^{ii}[\delta_{ii} - h_{ii}\langle X,\ \nu\rangle] \\
& - \frac{\langle X,\ e_1\rangle}{\langle X,\ \nu\rangle} F^{ii}\nabla_1 h_{ii} - \frac{F^{ii}h_{ii}}{\langle X,\ \nu\rangle} + F^{ii}h_{ii}^2.
\end{aligned}$$

Differentiating equation (2.6) with respect to e_1,

$$\begin{aligned}
F^{ii}\nabla_1 h_{ii} &= p\langle X,\ \nu\rangle^{p-1}\phi\nabla_1\langle X,\ \nu\rangle + \langle X,\ \nu\rangle^p\nabla_1\phi \\
&= p\langle X,\ \nu\rangle^{p-1}\phi h_{11}\langle X,\ e_1\rangle + \langle X,\ \nu\rangle^p\nabla_1\phi.
\end{aligned}$$

Thus we obtained

$$\begin{aligned}
0 \geqslant\ & [\gamma'' + (\gamma')^2][|X|^2 - \langle X,\ \nu\rangle^2]F^{11} + \gamma'\sum_{i=1}^n F^{ii} - \gamma' t\phi\langle X,\ \nu\rangle^{(p+1)} \\
& - [|X|^2 - \langle X,\ \nu\rangle^2]p\phi\langle X,\ \nu\rangle^{(p-1)}\gamma' - \langle X,\ e_1\rangle\langle X,\ \nu\rangle^{(p-1)}\nabla_1\phi \\
& - t\langle X,\ \nu\rangle^{(p-1)}\phi + F^{ii}h_{ii}^2.
\end{aligned}$$

So we have

$$\begin{aligned}
&[\gamma'' + (\gamma')^2]\langle X,\ \nu\rangle^2 F^{11} + (t-p)\gamma'\phi\langle X,\ \nu\rangle^{(p+1)} \\
&\quad + \left[|X|^2 p\phi\gamma' + \langle X,\ e_1\rangle\nabla_1\phi + t\phi\right]\langle X,\ \nu\rangle^{(p-1)} \\
&\geqslant [\gamma'' + (\gamma')^2]|X|^2 F^{11} + \gamma'\sum_{i=1}^n F^{ii}.
\end{aligned} \tag{2.9}$$

We may assume $\langle X,\ \nu\rangle^2 \leqslant C|X|^2$ for some $C>0$, otherwise the lemma holds. we claim firstly that

$$(t-p)\gamma'\phi\langle X,\ \nu\rangle^{(p+1)} + \left[|X|^2 p\phi\gamma' + \langle X,\ e_1\rangle\nabla_1\phi + t\phi\right]\langle X,\ \nu\rangle^{(p-1)} \leqslant 0, \tag{2.10}$$

by taking $\gamma(s)$ properly. We check it by three case of $p<0$, or $p>t$, and $0<p\leqslant t$

We taking

$$\gamma(s) = \frac{\alpha p}{s},$$

for $\alpha>0$ is large enough,

Case (i) $p<0$, **or** $p>t$: Assuming

$$\langle X,\ \nu\rangle^2 \leqslant \frac{p}{4(p-t)}|X|^2.$$

Then

$$\begin{aligned}
&(t-p)\gamma'\phi\langle X,\ \nu\rangle^{(p+1)} + |X|^2 p\phi\gamma'\langle X,\ \nu\rangle^{(p-1)}\\
&\quad= \phi\langle X,\ \nu\rangle^{(p-1)}\left[(t-p)\gamma'\langle X,\ \nu\rangle^2 + p|X|^2\gamma'\right]\\
&\quad= -\frac{4\alpha\phi\langle X,\ \nu\rangle^{(p-1)}}{|X|^2}\left[p^2 + (t-p)\frac{p\langle X,\ \nu\rangle^2}{|X|^2}\right]\\
&\quad\leqslant -\frac{p^2\alpha\phi\langle X,\ \nu\rangle^{(p-1)}}{|X|^2},
\end{aligned}$$

which implies (2.10) for $\alpha>0$ large enough.

Case (ii) $0<p\leqslant t$: we have

$$(t-p)\gamma'\phi\langle X,\ \nu\rangle^{(p+1)} \leqslant 0,$$

$$|X|^2 p\phi\gamma' + \langle X,\ e_1\rangle\nabla_1\phi + t\phi = -\frac{4\alpha p^2}{|X|^2} + \langle X,\ e_1\rangle\nabla_1\phi + t\phi \leqslant 0,$$

which imply inequality (2.10).

Combing (2.9) with (2.10),

$$\begin{aligned}
&[\gamma'' + (\gamma')^2]\langle X,\ \nu\rangle^2 F^{11}\\
&\quad\geqslant [\gamma'' + (\gamma')^2]|X|^2 F^{11} + \gamma'\sum_{i=1}^n F^{ii}.
\end{aligned}\tag{2.11}$$

On the other hand,

$$\begin{aligned}
&[\gamma'' + (\gamma')^2]|X|^2 F^{11} + \gamma'\sum_{i=1}^n F^{ii}\\
&\quad= [\frac{16\alpha p}{|X|^6} + \frac{16\alpha^2 p^2}{|X|^8}]|X|^2 F^{11} - \frac{4\alpha p}{|X|^4}\sum_{i=1}^n F^{ii}\\
&\quad\geqslant C_0 F^{11},
\end{aligned}\tag{2.12}$$

which is from $h_{11} \leqslant 0$ and then $F^{11} \geqslant c_0 \sum\limits_{i=1}^n F^{ii}$. We test the case of $F = \frac{\sigma_k}{\sigma_l}$($F = \sigma_k$ see [9]), from (25) in [15]

$$F^{11} \geqslant C(n,l,k)\frac{\sigma_{k-1}(\lambda|1)}{\sigma_l} \geqslant C(n,l,k)\frac{\sigma_{k-1}(\lambda)}{\sigma_l},$$

and

$$\begin{aligned} -\sum_{i=1}^n F^{ii} &= -\frac{(n-k+1)\sigma_{k-1}}{\sigma_l} + \frac{(n-l+1)\sigma_k\sigma_{l-1}}{\sigma_l^2} \\ &\geqslant -\frac{(n-k+1)\sigma_{k-1}}{\sigma_l}. \end{aligned}$$

Thus $\langle X,\ \nu\rangle \geqslant C$ is from (2.11) and (2.12), that is to say there exists a constsnt that depends only on n, t, $\min_{\mathbb{S}^n} \phi$, $|\phi|_{C^1}$ such that

$$\max_{\mathbb{S}^n} |\nabla\rho| \leqslant C.$$

□

3. The Important C^2 Estimates.

The following lemma is key in our proof for C^2 estimate, which is from Guan-Li-Li's important lemma[5, 10].

Lemma 3.1. *For any $\alpha > 0$, one has the following inequality*

$$\sigma_k^{ij,mq}\nabla_s h_{ij}\nabla_s h_{mq} \leqslant \sigma_k\left(\frac{\nabla_s\sigma_k}{\sigma_k} - \frac{\nabla_s\sigma_1}{\sigma_1}\right)\left((\alpha+1)\frac{\nabla_s\sigma_k}{\sigma_k} - (\alpha-1)\frac{\nabla_s\sigma_1}{\sigma_1}\right)$$

Proof. From Krylov[14], for any $\alpha > 0$, $\left(\frac{\sigma_1}{\sigma_k}\right)^\alpha$ is convex on Γ_k, thus,

$$\begin{aligned} 0 &\leqslant \left[\left(\frac{\sigma_1}{\sigma_k}\right)^\alpha\right]^{ij,mq} \nabla_s h_{ij}\nabla_s h_{mq} \\ &= \alpha\left(\frac{\sigma_1}{\sigma_k}\right)^{\alpha-1} 2\left(\frac{\sigma_1|\nabla\sigma_k|^2}{\sigma_k^3} - \frac{\langle\nabla\sigma_1, \nabla\sigma_k\rangle}{\sigma_k^2}\right) \\ &\quad -\alpha\left(\frac{\sigma_1}{\sigma_k}\right)^{\alpha-1} \frac{\sigma_1}{\sigma_k^2}\sigma_k^{ij,mq}\nabla_s h_{ij}\nabla_s h_{mq} \\ &\quad +\alpha(\alpha-1)\left(\frac{\sigma_1}{\sigma_k}\right)^{\alpha-2}\left(\frac{\nabla\sigma_1}{\sigma_k} - \frac{\sigma_1\nabla\sigma_k}{\sigma_k^2}\right)^2. \end{aligned}$$

This implies

$$\begin{aligned}
0 \leqslant\ & \frac{2}{\sigma_1}\left(\frac{\sigma_1|\nabla\sigma_k|^2}{\sigma_k} - \langle\nabla\sigma_1, \nabla\sigma_k\rangle\right) \\
& -\sigma_k^{ij,mq}\nabla_s h_{ij}\nabla_s h_{mq} \\
& +(\alpha-1)\left(\frac{\sigma_1^2}{\sigma_k}\right)^{-1}\left(\nabla\sigma_1 - \frac{\sigma_1\nabla\sigma_k}{\sigma_k}\right)^2 \\
=\ & \frac{(\alpha+1)|\nabla\sigma_k|^2}{\sigma_k} - 2\alpha\frac{\langle\nabla\sigma_1, \nabla\sigma_k\rangle}{\sigma_1} \\
& -\sigma_k^{ij,mq}\nabla_s h_{ij}\nabla_s h_{mq} \\
& +(\alpha-1)\sigma_k\left(\frac{|\nabla\sigma_1|}{\sigma_1}\right)^2.
\end{aligned}$$

we have proved this lemma. □

Theorem 3.2. *Let $\phi(X)$ be a C^2 positive function on $\mathbb{M}$, if $\mathbb{M}$ is an admissible solution of* (1.3)*, we have the following estimates*

$$\sigma_1(A) \leqslant C(n, k, min_{\mathbb{M}} f, \|f\|_{C^2}). \tag{3.1}$$

Proof. Considering

$$F(A) = \sigma_k(A) = \langle X,\ \nu\rangle^p\phi(X), \tag{3.2}$$

we denote $\langle X,\ \nu\rangle$ by u in what follows.

Taking test function $\frac{\sigma_1}{u}$, then at its maximal point P

$$\nabla_i\left(\ln\frac{\sigma_1}{u}\right) = 0,$$

and

$$\begin{aligned}
0 \geqslant\ & F^{ij}\nabla_i\nabla_j\left(\ln\frac{\sigma_1}{u}\right) \\
=\ & F^{ij}\left[\frac{\nabla_i\nabla_j\sigma_1}{\sigma_1} - \frac{\nabla_i\sigma_1\nabla_j\sigma_1}{\sigma_1^2} - \frac{\nabla_i\nabla_j u}{u} + \frac{\nabla_i u\nabla_j u}{u^2}\right] \\
=\ & \frac{1}{\sigma_1}F^{ij}\nabla_i\nabla_j\sigma_1 - \frac{1}{u}F^{ij}\nabla_i\nabla_j u
\end{aligned} \tag{3.3}$$

which is equivalent to

$$0 \geqslant \frac{1}{u}F^{ij}\nabla_i\nabla_j\sigma_1 - \frac{1}{u}\left(\frac{\sigma_1}{u}\right)F^{ij}\nabla_i\nabla_j u. \tag{3.4}$$

On the other hand, we have

$$\begin{aligned}
-\frac{1}{u}\left(\frac{\sigma_1}{u}\right)F^{ij}\nabla_i\nabla_j u =\ & -\frac{1}{u}\left(\frac{\sigma_1}{u}\right)F^{ij}\left[\nabla_m h_{ij}\langle X, X_m\rangle + h_{ij} - h_{im}h_{mj}u\right] \\
=\ & -\frac{1}{u}\left(\frac{\sigma_1}{u}\right)\nabla_m F\langle X, X_m\rangle - kF\frac{\sigma_1}{u^2} + \left(\frac{\sigma_1}{u}\right)F^{ij}h_{im}h_{mj}.
\end{aligned} \tag{3.5}$$

We also compute the following by lemma 2.1,

$$\begin{aligned}
\frac{1}{u}F^{ij}\nabla_i\nabla_j\sigma_1 &= \frac{1}{u}F^{ij}\nabla_s\nabla_s h_{ij} + \frac{kF}{u}|A|^2 - \frac{1}{u}F^{ij}h_{im}h_{mj}\sigma_1 \\
&= \frac{1}{u}\triangle F - \frac{1}{u}F^{ij;mq}\nabla_s h_{ij}\nabla_s h_{mq} + \frac{kF}{u}|A|^2 - \frac{1}{u}F^{ij}h_{im}h_{mj}\sigma_1 \\
&= \frac{1}{u}[u^p\triangle\phi + 2pu^{p-1}\langle\nabla\phi,\nabla u\rangle] + p\phi u^{p-2}\triangle u + p(p-1)u^{p-3}|\nabla u|^2 \\
&\quad -\frac{1}{u}F^{ij;mq}\nabla_s h_{ij}\nabla_s h_{mq} + \frac{kF}{u}|A|^2 - \left(\frac{\sigma_1}{u}\right)F^{ij}h_{im}h_{mj} \\
&= \frac{1}{u}[u^p\triangle\phi + 2pu^{p-1}\langle\nabla\phi,\nabla u\rangle] + p\phi u^{p-2}[\nabla_m\sigma_1\langle X,\ X_m\rangle + g] \\
&\quad +p(p-1)u^{p-3}|\nabla u|^2 - \frac{1}{u}F^{ij;mq}\nabla_s h_{ij}\nabla_s h_{mq} \\
&\quad +\frac{(k-p)F}{u}|A|^2 - \left(\frac{\sigma_1}{u}\right)F^{ij}h_{im}h_{mj}.
\end{aligned}\tag{3.6}$$

Then , combing (3.4) with (3.5), (3.6),

$$\begin{aligned}
0 &\geqslant -\frac{1}{u}F^{ij;mq}\nabla_s h_{ij}\nabla_s h_{mq} + (k-p)\phi u^{p-1}|A|^2 \\
&\quad +p(p-1)\phi u^{p-3}|\nabla u|^2 - C\sigma_1 - C.
\end{aligned}\tag{3.7}$$

Then with lemma 3.1, and $\frac{(\sigma_k)_s}{\sigma_k} = p\frac{u_s}{u} + \frac{\phi_s}{\phi}$, $\frac{(\sigma_1)_s}{\sigma_1} = \frac{u_s}{u}$,

$$0 \geqslant (k-p)\phi u^{p-1}|A|^2 + (p-1)[p-(\alpha+1)p+(\alpha-1)]\phi u^{p-3}|\nabla u|^2 - C\sigma_1 - C,$$

one has the C^2 estimate if $(p-1)[p-(\alpha+1)p+(\alpha-1)] \geqslant 0$, which is satisfied by taking $p \leqslant 1$ and $\alpha = \frac{1}{1-p} > 0$. □

From the above certificate process, we know the key point for proving C^2 estimates is the concavity of $[\frac{\sigma_k}{\sigma_1}]^{\frac{1}{k-1}}$, our lemma here has used the convexity of $\left(\frac{\sigma_1}{\sigma_k}\right)^\alpha$ on Γ_k.

4. Proof of theorem 1.2

4.1. **Proof of theorem 1.2.** We use the method of continuity to prove theorem 1.2. For any positive function $\phi(X) \in \mathbb{M}$, and $t \in [0,1]$, setting

$$\phi_t(X) = 1 - t + t\phi(X),$$

we consider a family of equations

$$\sigma_k(A) = \langle X,\ \nu\rangle^p\phi_t(X). \tag{4.1}$$

and

$$I = \{t \in [0,1] |\ \textit{Equation (4.1) has a smoothe admissible solution}\}.$$

For $t = 0$, $X = (C_n^k)^{-\frac{1}{k-p}}$ is a solution of (4.1), i.e. I is not empty. Moreover, The a prior estimates lemma 2.2, lemma 2.2 and theorem 3.2 and Evans-Krylov theorem imply the closeness of I. we prove the following proposition that is to say I is open.

Proposition 4.1. *Assume $F(\lambda)$ and $\phi(x,\rho,\nabla\rho)$ satisfying homogeneity property:*

$$F(\lambda(t\rho)) = F(\frac{\lambda(\rho)}{t}) \tag{4.2}$$

$$\phi(x,t\rho,t\nabla\rho) = t^s\phi(x,\rho,\nabla\rho). \tag{4.3}$$

Then the linearized operator L of $F(\lambda) = \phi(x,\rho,\nabla\rho)$ has no non zero kernel, which is from lemma 2.5 in [9].

Lastly, the uniqueness result of such problem is same as lemma 2.4 in [9]. We have complete the proof of theorem 1.2.

5. SOME DISCUSS ABOUT IVOCHKINA'S PROBLEM

Ivochkina[12, 13] had considered the generalized type of curvature equation, see also [2]v and their references,

$$\sigma_k(\lambda) = \phi(x,g,Dg), \tag{5.1}$$

where $\lambda = (\lambda_1, \cdots, \lambda_n)$ is the principal curvatures of the graph

$$\mathbb{M} = \{(x,g(x))|x\in\Omega\}.$$

In Ivochkina's notation, $\phi(x,g,Dg) = \frac{f(x,g,Dg)}{(1+|Dg|^2)^{\frac{k}{2}}}$, she need her condition (1.5) in [12](see also (8.28) in [13]) to do C^2 estimate. An example is

$$f(x,g,Dg) = H(x,g)(1+|Dg|^2)^s,$$

for $s \geqslant \frac{k}{2}$. Please note that there is a misprint at page 334 in [12] for that example, that is no her (1.5) for $s \geqslant 1/2$. The author want to thank for Ivochkina's mention for that.

Here we give a C^2 estimate for a special case $\phi(x,u,Du)$, we consider the example which is similar to (2.6) and (2.7) in [20].

$$\sigma_k(A) = \frac{H(x,g)}{(1+|Dg|^2)^{\frac{q}{2}}}, \tag{5.2}$$

for this example, Ivochkina need

$$s = \frac{k-q}{2} \geqslant \frac{k}{2} \Leftrightarrow q \leqslant 0.$$

Proof of Theorem 1.3:

Proof. As [19], taking a local orthonormal frame field $e_1, \cdots, e_n$ defined on $\mathbb{M} = \{(x,u(x))|x\in\Omega\}$ in a neighbourhood of the point at which we are computing and the upward unit normal vector field is

$$\nu = \frac{(-Dg,1)}{\sqrt{1+|Dg|^2}} \triangleq \frac{(-Dg,1)}{w}.$$

We consider test function $\widehat{W} = w\sigma_1 h(\frac{w^2}{2})$ for $h(t) > 0$ to be determined, and it attain its interior maximum at X_0.

By Lemma 2.1 in [22],

$$\nabla_i\nabla_j w = wh_{im}h_{mj} + 2\frac{\nabla_i w\nabla_j w}{w} + w^2\langle\nabla h_{ij},\ E_{n+1}\rangle, \tag{5.3}$$

where $A=[h_{ij}]$, E_{n+1} is the $n+1$-st coordinate vector in $\mathbb{R}^{n+1}$.

$$0=\nabla_i \ln \widehat{W}=\frac{\nabla_i w}{w}+\frac{h'w\nabla_i w}{h}+\frac{\nabla_i\sigma_1}{\sigma_1}, \tag{5.4}$$

and

$$\begin{aligned} 0 \ &\geqslant \ F^{ij}\nabla_i\nabla_j \ln \widehat{W}\\ &= \ \left(\frac{1}{w}+\frac{h'w}{h}\right)F^{ij}\nabla_i\nabla_j w+F^{ij}\frac{\nabla_i\nabla_j\sigma_1}{\sigma_1}\\ &\quad -\left(\frac{2}{w^2}-\frac{w^2h''}{h}+\frac{2h'^2w^2}{h^2}+\frac{h'}{h}\right)F^{ij}\nabla_i w\nabla_j w\\ &= \ \left(1+\frac{h'w^2}{h}\right)F^{ij}h_{mi}h_{mj}+F^{ij}\frac{\nabla_i\nabla_j\sigma_1}{\sigma_1}\\ &\quad +\left(\frac{w^2h''}{h}-\frac{2h'^2w^2}{h^2}+\frac{h'}{h}\right)F^{ij}\nabla_i w\nabla_j w\\ &\quad +(w+\frac{h'w^3}{h})\langle\nabla F,\ E_{n+1}\rangle. \end{aligned} \tag{5.5}$$

By (2.9) in [22],

$$F^{ij}\frac{\nabla_i\nabla_j\sigma_1}{\sigma_1}=-\frac{F^{ij,pq}\nabla_\alpha h_{ij}\nabla_\alpha h_{pq}}{\sigma_1}+F^{ij}h_{ij}\frac{|A|^2}{\sigma_1}-F^{ij}h_{mi}h_{mj}+\frac{\triangle F}{\sigma_1},$$

this combines with (5.5), one has

$$\begin{aligned} 0 \ &\geqslant \ \frac{h'w^2}{h}F^{ij}h_{mi}h_{mj}-\frac{F^{ij,pq}\nabla_\alpha h_{ij}\nabla_\alpha h_{pq}}{\sigma_1}+F^{ij}h_{ij}\frac{|A|^2}{\sigma_1}\\ &\quad +\left(\frac{w^2h''}{h}-\frac{2h'^2w^2}{h^2}+\frac{h'}{h}\right)F^{ij}\nabla_i w\nabla_j w\\ &\quad +(w+\frac{h'w^3}{h})\langle\nabla F,\ E_{n+1}\rangle+\frac{\triangle F}{\sigma_1}. \end{aligned}$$

Taking

$$h(t)=e^{\frac{t}{2\sup w^2}},$$

we have

$$\frac{w^2h''}{h}-\frac{2h'^2w^2}{h^2}+\frac{h'}{h}\geqslant 0.$$

Thus

$$\begin{aligned} 0 \ &\geqslant \ \frac{h'w^2}{h}F^{ij}h_{mi}h_{mj}-\frac{F^{ij,pq}\nabla_\alpha h_{ij}\nabla_\alpha h_{pq}}{\sigma_1}+F^{ij}h_{ij}\frac{|A|^2}{\sigma_1}\\ &\quad +(w+\frac{h'w^3}{h})\langle\nabla F,\ E_{n+1}\rangle+\frac{\triangle F}{\sigma_1}. \end{aligned} \tag{5.6}$$

From (5.2), we set

$$F=\sigma_k(A)=\frac{H(x,g)}{(1+|Dg|^2)^{\frac{q}{2}}},$$

where $H(x,g)$ does not impact the process of proof in the follows, we may consider the following special case for simplifying the denotation,

$$F = \sigma_k(A) = (1+|Dg|^2)^{-\frac{q}{2}} = w^{-q}, \tag{5.7}$$

inserting this into (5.6), and noticing (5.3) and (5.4),

$$\begin{aligned} 0 \quad \geqslant \quad & \frac{h' w^2}{h} F^{ij} h_{mi} h_{mj} - \frac{F^{ij,pq}\nabla_\alpha h_{ij} \nabla_\alpha h_{pq}}{\sigma_1} + (k-q)F\frac{|A|^2}{\sigma_1} \\ & + \frac{(q^2-q)w^{-(2+q)}|\nabla w|^2}{\sigma_1} - C, \end{aligned} \tag{5.8}$$

for $\sigma_1 >> 1$.

On the other hand, lemma 3.1 and (5.7) implies

$$\begin{aligned} -\frac{F^{ij,pq}\nabla_\alpha h_{ij}\nabla_\alpha h_{pq}}{\sigma_1} \quad \geqslant \quad & \frac{w^{-(2+q)}|\nabla w|^2}{\sigma_1}\{-(q-1)^2\alpha - q^2 + 1 \\ & + [(q-1)(2\alpha-1)+1+q]\,\frac{h'}{h}w^2 \\ & +(1-\alpha)\frac{h'^2}{h^2}w^4\}. \end{aligned} \tag{5.9}$$

Taking $\alpha = \frac{1}{1-q}$, if $q<1$ and $0<\alpha<1$, if $q=1$, then

$$-\frac{F^{ij,pq}\nabla_\alpha h_{ij}\nabla_\alpha h_{pq}}{\sigma_1} + \frac{(q^2-q)w^{-(2+q)}|\nabla w|^2}{\sigma_1} \geqslant 0. \tag{5.10}$$

Lastly, the C^2 estimates (1.7) is from (5.8) and (5.10).

□

Remark 5.1. $\frac{h'w^2}{h}F^{ij}h_{mi}h_{mj}$ *is a good term for our estimate in* (5.8). *Ivochkina has used it in* [12, 13]. *one may use it to control the term like* $-\frac{w^{-(2+q)}|\nabla w|^2}{\sigma_1}$ *and refine theorem 1.3.*

Remark 5.2. *Takimoto*[20] *had used a priori estimates of the second derivatives of* u *in his (2.6) and (2.7) for* $1 \leqslant q \leqslant k-1$ *at page 368. So his result is incomplete.*

Remark 5.3. *In the end, an interesting problem is what we can generalize Ivochkinas'* C^2 *estimates in* [12, 13] *to the following quotient curvature equations?*

$$\frac{\sigma_k}{\sigma_l}(A) = H(x,g)w^{-q}, \tag{5.11}$$

where $0 \leqq l < k \leqq n$. *Of course, it is also interesting for generalized Guan-Li-Li's results to quotient equations.*

Remark 5.4. *More recently, we have found that Chuanqiang Chen had obtained* C^2 *estimates for our problem in case of* $k=2$ *by using different methods :"Chuanqiang Chen. A Minimal Value Problem and the Prescribed* σ_2 *Curvature Measure Problem, arXiv:1104.4283"*

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